

Amogh Gupta, Sylvia Jin, Aekus Bhathal, Abinav Routhu, Debayan Bandyopadhyay
Roast us here: <https://tinyurl.com/csm70-feedback20>

Computability

1. Say that we have a program M that decides whether any input program halts as long as it prints out the string "ABC" as the first operation that it carries out. Can such a program exist? Prove your answer.

2. (a) Is it possible to write a program that takes a natural number n as input, and finds the shortest arithmetic formula which computes n ? For the purpose of this question, a formula is a sequence consisting of some valid combination of (decimal) digits, standard binary operators (+, ×, the “^” operator that raises to a power), and parentheses. We define the length of a formula as the number of characters in the formula. Specifically, each operator, decimal digit, or parentheses counts as one character.

(Hint: Think about whether it’s possible to enumerate the set of possible arithmetic formulas. How would you know when to stop?)

- (b) Now say you wish to write a program that, given a natural number input n , finds another program (e.g. in Java or C) which prints out n . The discovered program should have the minimum execution-time-plus-length of all the programs that print n . Execution time is measured by the number of CPU instructions executed, while “length” is the number of characters in the source code. Can this be done?

(Hint: Is it possible to tell whether a program halts on a given input within t steps? What can you say about the execution-time-plus-length of the program if you know that it does not halt within t steps?)

3. (a) Explain why the notion of the "smallest positive integer that cannot be defined in under 280 characters" is paradoxical.
- (b) Prove that for any length n , there is at least one string of bits that cannot be compressed to less than n bits.
- (c) Say you have a program K that outputs the Kolmogorov complexity of any input string. Under the assumption that you can use such a program K as a subroutine, design another program P that takes an integer n as input, and outputs the length- n binary string with the highest Kolmogorov complexity. If there is more than one string with the highest complexity, output the one that comes first lexicographically.
- (d) Let's say you compile the program P you just wrote and get an m bit executable, for some $m \in \mathbb{N}$ (i.e. the program P can be represented in m bits). Prove that the program P (and consequently the program K) cannot exist.
- (Hint: Consider what happens when P is given a very large input n .)