6

Counting, Combinatorics, Countability

Computer Science Mentors 70

Mar 08 - Mar 12
Spring 2021

Amogh Gupta, Sylvia Jin, Aekus Bhathal, Abinav Routhu, Debayan Bandyopadhyay
Roast us here: `https://tinyurl.com/csm70-feedback20`

1 Counting

1. How many length 7 bitstrings have more zeroes than ones?

Solution: There are a total of 2^7 bitstrings, since each bit can be either 0 or 1. Since the bitstring is of odd length, it needs to either have more zeroes than ones or vice versa. One can take a bitstring that has more zeroes than ones, and flip all of the bits so that there are now more ones than zeroes, so there are an equal amount of either type. (In other words, there is a bijection between the set of length 7 bitstrings that have more ones and the set of length 7 bitstrings that have more zeros. The two sets are disjoint and make up the whole set of length 7 bitstrings.) $2^7/2 = 64$.

2. How many length 8 bitstrings have more zeroes than ones?

Solution: Similar approach as above, but we must also consider the bitstrings that have an equal number of 0s and 1s. (There's still a bijection, but now there are bitstrings that don't fit into either set.) There are $\binom{8}{4}$ of these as we pick 4 positions to place the 1s in, and the rest of the bitstrings either have more zeroes than ones or vice versa. $(2^8 - \binom{8}{4})/2 = 93$.

3. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order. (Source: 2020 AIME I)

Solution: We can deal with one of the cases first, say the ascending case, and multiply by 2 since we know that for each ascending case there is a unique corresponding decreasing case. All valid sequences can be constructed by taking all numbers but one, order them in an increasing sequence, and then insert the final number, say some number k, at any location. This is valid because k can be deleted and the sequence is still ascending. Since there are 6 possible values of k, and 6 locations that they can be inserted into, we have a total of 6×6 choices.
However, we have overcounted here. Let us visualize by expanding on the cases of inserting 1, 2, and 3:

inserting 1	inserting 2	inserting 3
123456	213456	312456
213456	123456	132456
231456	132456	123456
234156	134256	124356
234516	134526	124536
234561	134562	124563

Each time, there are two sequences that are shared with each adjacent list (and one of these sequences is always 123456). This is because, for each pair $\{k, k+1\}$, we want to include the sequences $\ldots k, k+1, \ldots$ and $\ldots k+1, k, \ldots$. These sequences will both appear twice, once when k and $k+1$ each are being inserted, but we want to only keep one of each. Thus, we subtract 2 for every adjacent pair of numbers, of which there are 5. We can also think about it as taking out the extra five 123456's that we end up with, and also taking out the repeat instance of having consecutive digits next to each other.
After accounting for overcounting and doubling the count since we can have decreasing sequences as well, the answer

should be $(6 \times 6 - 10) \times 2 = 52$.

4. How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

Solution: We can think of this in terms of stars and bars. We have two bars between the variables x, y, and z, and our stars are the 10 1's we have to distribute among them. Since all variables must be positive integers, x, y, and z will each be at least 1. So, we have 7 1's left to distribute. So we have 7 stars, 2 bars. Answer = $\binom{7+2}{2} = \binom{9}{2} = 36$.

5. How many solutions does $x_1 + x_2 + x_3 + x_4 = 20$ have, if all variables must be non-negative integers, $x_1 < 8$, and $x_2 < 8$?

Solution: We can think of this in terms of stars and bars. We have three bars between the four variables x_1, x_2, x_3, and x_4, and our stars are the 20 1's we have to distribute among them. But given the the restrictions $x_1 < 8$ and $x_2 < 8$, we need to use Inclusion-exclusion and subtract the cases where $x_1 > 7$ or $x_2 > 7$ from the number of solutions.

Number of ways to have $x_1 + x_2 + x_3 + x_4 = 20$ without restrictions on variables, we have 20 1's left to distribute. So we have 20 stars, 3 bars. Answer = $\binom{20+3}{3} = \binom{23}{3} = 1771$.

Now suppose $x_1 > 7$ Then we have x_1 is atleast 8 and we are left with only $20 - 8 = 12$ stars to distribute and 3 bars. Answer = $\binom{12+3}{3} = \binom{15}{3} = 455$. We have a similar case when $x_2 > 7$

Now for the case when $x_1 > 7$ and $x_2 > 7$ we are left with only $20 - 8 - 8 = 4$ stars to distribute and 3 bars. Answer = $\binom{4+3}{3} = \binom{7}{3} = 35$.

Then finally using inclusion- exclusion, we get Answer = $1771 - 2\dot{4}55 + 35 = 895$

6. Consider a little game. Alice and Bob arrange 8 coins in a circle with alternating faces up (i.e. Heads, Tails, Heads, Tails, ...), and they each take turns flipping coins according to the following rules:

Alice chooses any coin with heads facing up and flips it. Then, she sets the coin adjacent to it in the clockwise direction to heads.
Bob chooses any coin with tails facing up and flips it. Then, he sets the coin two spots away (index $i + 2$) in the clockwise direction to tails.

Assume Alice always plays the first move.

(a) How many distinct ways are there for Alice and Bob to play the first two moves (hint: how many unique tuples (i, j) exist where i is the index of the first coin being flipped and j is the index of the second).

Solution: 16

There are initially 4 coins with heads, meaning i can take on 4 possible values. Now, no matter what coin Alice picks, the coin adjacent to it in the clockwise direction is always set to tails. Hence, Alice is flipping one coin from heads to tails and one coin from tails to heads, so the total number of Heads and Tails after her turn are 4 each.

Now, on Bob's turn, j can take on 4 possible values, since there are 4 tails. Hence, by the First Rule of Counting, there are a total of $4 \cdot 4 = 16$ ways of selecting the indices for Alice and Bob's first two moves.

(b) Suppose the game ends when there are no legal moves for the next player. Prove that no matter what Alice and Bob play, the game will never end.

Solution: If there are no legal moves for a given player, then all the coins must have the same face (that is, either all heads or all tails). We prove the contrapositive of this statement: if at every turn, there is at least one head and at least one tail, then there will always be a legal move.

First, we notice that at every turn, the number of heads or tails increases by 0 or 1. Hence, before there are no heads or no tail, there must be a turn where there is one head or one tail.

Case 1: If there is one head, then either it is Bob's turn, and the number of heads either stays the same or increases, or it is Alice's turn and she flips it to a tail. However, Alice must set the coin adjacent to it to a head, so the number of heads remains 1.

Case 2: If there is one tail, then either it is Alice's turn, and the number of tails either stays the same and increases, or it is Bob's turn and he flips the tail to a head. Now, Bob would have to set the coin two places clockwise to a tail, meaning that the number of tails remains 1.

Since there is always at least one tail and one head, the game can never end.

(c) Use parts (a) and (b) to provide an upper bound to the number of distinct ways for Alice and Bob to play the first k moves. Assume k is even.
Hint: to find an upper bound, think about the maximum number of choices Alice/Bob can make at each step.

Solution:

Solution 1: $4^2 \cdot 7^{k-2}$

Using part (a), the first two moves can be played in 4^2 ways. From part (b), we determined that the game never ends, meaning there is always at least one head or tail on the board. Hence, Alice or Bob can choose from at most 7 coins on every subsequent turn. Using the First Rule of Counting, thus, we provide an upper bound of $4^2 \cdot 7^{k-2}$

Solution 2: $4^2 \cdot 20^{\frac{k}{2}-1}$

We can actually provide a much better bound. Since at every turn, we notice that if Alice has h heads to flip from, then after her turn, the number of tails for Bob to pick from is either $8 - h$ or $8 - h + 1$. Hence, the number of moves in any pair of turns $(k, k+1)$ is at most $h(8 - h + 1)$ by the First Rule of Counting. This function is maximized when $h = 4$ or $h = 5$, giving us an upper bound of 20 moves for any pair of turns. Since we proved that the game never ends, there is no line of play that terminates within the first k moves. Hence, we can once again use the First Rule of Counting to upper bound the number of moves in turns $3 \ldots k$ by $20^{\frac{k-2}{2}}$. Since there are 16 ways of arranging the first two moves, we apply the first rule of counting once again, giving us an upper bound of $4^2 \cdot 20^{\frac{k}{2}-1}$

(d) How many distinct arrangements are there for the coins after 2 moves have been played? Assume that any arrangement of coins that can be reached by rotating all the coins by $1 \leq i \leq 7$ indices is equivalent.

Solution: 3

Suppose the indices of the heads before Alice makes her move are 1, 3, 5, and 7. Now, after she makes her move, the indices of the heads could be:

2,3,5,7
1,4,5,7
1,3,6,7
1,3,5,8

We also know that the indices are unique modulo 8 (rotating the entire circle by 8 indices leaves all indices at their original positions). Hence, rotating the second configuration by 6 indices clockwise, the third configuration by 4 and the fourth configuration by 2 all modulo 8 gives us:

$(2, 3, 5, 7) \pmod 8$
$(7, 2, 3, 5) \pmod 8$
$(5, 7, 2, 3) \pmod 8$
$(3, 5, 7, 2) \pmod 8$

These are all equivalent arrangements though, since all the heads are at the same indices.

Now, we can without loss of generality consider the arrangement $THHTHTHT$. After Bob takes his turn, the resulting arrangements are

$HHTTHTHT$
$THHHHTHT$
$THHTHHHT$
$TTHTHTHH$

Now, we notice that the fourth arrangement can be rotated by 2 indices to reach the first arrangement. Hence, there are only 3 unique arrangements after 2 turns.

2 Combinatorial Proofs

1. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

   Which of the following are valid ways of counting the number of squares in an $n \times n$ grid?

   (a) In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.

   (b) We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n-1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n-2$ squares on the hypotenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

   (c) Take the $(n-1) \times (n-1)$ subgrid that is the upper-left corner of this grid. This subgrid has $n-1$ rows, each of which has $n-1$ squares, so this part contributes $(n-1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n-1)^2 + 2n - 1$ squares in an $n \times n$ grid.

   (d) First, we peel off the leftmost column, and topmost row, removing exactly $2n-1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n-1)-1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n-1) + (2n-3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

   Solution: All of the above methods are valid ways of counting!

2. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

   Solution: Choose a team of k players where one of the players is the captain.
   LHS: Pick a team with k players. This is $\binom{n}{k}$. Then make one of the players the captain. There are k options for the captain so we get $k \times \binom{n}{k}$.
   RHS: Pick the captain. There are n choices for the captain. Now pick the last $k-1$ players on the team. There are now $n-1$ people to choose from. So we get $n \times \binom{n-1}{k-1}$.

3. Prove $\binom{n}{a}a(n-a) = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

   Solution: Suppose that you have a group of n players and want to pick a team of a with a captain, as well as a reserve player from the remaining $n-a$ players.
   LHS: Number of ways to pick a team of a of these players ($\binom{n}{a}$ ways), designate one member of the team as captain (a ways), and then pick one reserve player from the remaining $n-a$ people ($n-a$ ways).
   RHS: The right-hand side is the number of ways to pick the captain (n ways), then the reserve player ($n-1$ ways), and then the other $a-1$ members of the team ($\binom{n-2}{a-1}$ ways).

3 Countability

1. Show that for any positive integer n, an injective (one-to-one) function $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ must be a bijection.

> **Solution:** Since f is an injection, every element must send to a distinct element. Thus, the cardinality of the image of the function f must also be n, which is equal to the cardinality of the range. Thus, the range and the image must be the same set, so the function is a surjection. Thus, f is a bijection.

2. Find a bijection between $\mathbb{N}$ and the set of all integers congruent to $1 \mod n$, for a fixed n.

> **Solution:** The set of integers congruent to $1 \mod n$ is $A = \{1 + kn \mid k \in \mathbb{Z}\}$. Define $g : \mathbb{Z} \longrightarrow A$ by $g(x) = 1 + x \cdot n$; this is a bijection because it is clearly one-to-one, and is onto by the definition of A. We can combine this with the bijective mapping $f : \mathbb{N} \longrightarrow \mathbb{Z}$ from the notes, defined by $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{-(x+1)}{2} & \text{if } x \text{ is odd} \end{cases}$. Then $g \circ f$ is a function from $\mathbb{N}$ to A, which is a bijection since the composition of bijections is a bijection.

3. Are these sets countably infinite/uncountably infinite/finite? If finite, what is the order of the set? Reminder: A bit string is a sequence of digits where each digit corresponds to either a 1 (on) or a 0 (off).

   (a) Finite bit strings of length n.

   > **Solution:** Finite. There are 2 choices (0 or 1) for each bit, and n bits, so there are $2 \times 2 \times \ldots \times 2 = 2^n$ such bit strings.

   (b) All finite bit strings of length 1 to n.

   > **Solution:** Finite. By part (a), there are 2^1 bit strings of length 1, 2^2 of length 2, etc. Thus, there are $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$ such bit strings.

   (c) All finite bit strings

   > **Solution:** Countably infinite. We can list these strings as follows: $\{0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots\}$. This gives us a bijection with the (countable) natural numbers, so these are countably infinite.

   (d) All infinite bit strings

   > **Solution:** Uncountably infinite. We can construct a bijection between this set and the set of real numbers between 0 and 1 in binary – e.g. they are of the form $0.0110001010110\ldots$. By diagonalization, the set of real numbers between 0 and 1 is uncountably infinite; therefore, so is this set.

   (e) All finite or infinite bit strings.

   > **Solution:** Uncountably infinite. This is the union of a countably infinite set (part c) and an uncountably infinite set (part d), so it is uncountably infinite.

   (f) Suppose that we try to show that infinite bit strings are countable by induction. We show that for any positive integer n, a bit string of length n is countable. Why does this not work for infinite strings?

Solution: Induction applies only to finite values.

4. If S is countably infinite, is the power set $\mathcal{P}(S)$ finite, countably infinite, or uncountably infinite? Provide a proof for your answer. Reminder: the power set of a set is the set of all possible subsets of that set. Ex: $S = \{A, B\}, \mathcal{P}(S) = \{\{\}, \{A\}, \{B\}, \{A, B\}\}$

Solution: The power sets of a countably infinite set are uncountably infinite. There is a bijection between the set $2^{\mathbb{N}}$ and 2^S, as S and $\mathbb{N}$ have the same cardinality. The set $2^{\mathbb{N}}$ is uncountable. We prove this through contradiction. Assume the set $2^{\mathbb{N}}$ is countably infinite. This means we can list the subsets of $\mathbb{N}$ such that every subset is N_i for some i. We define another set $A = \{i \mid i \geq 0 \text{ and } i \notin N_i\}$ which contains integers i not part of N_i. But A is a subset of $\mathbb{N}$, so we must have $A = N_j$ for some j. This means that if $j \in N$, then $j \notin N$, and if $j \notin N$, then $j \in N$. This is a contradiction since j is either in N or not, so the set is not countably infinite.
Alternate solution (direct proof): because $2^{\mathbb{N}}$ is the set of all functions $f : \mathbb{N} \rightarrow \{0, 1\}$, prepending a decimal point to each sequence of 0-1 bits gives you the set of all numbers between 0 and 1 (like “decimals” but in binary), which is uncountably infinite by diagonalization.